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# Metric Ricci Curvature and Flow for $PL$ Manifolds

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## Abstract

We summarize here the main ideas and results of our papers [28], [14], as presented at the 2013 CIRM Meeting on Discrete curvature and we augment these by bringing up an application of one of our main results, namely to solving a problem regarding cube complexes.

## 1. INTRODUCTION

While curvature itself represents a classical notion going back at least to Newton, it was traditionally restricted to the firmly established context of smooth (at least twice differentiable) curves and surfaces. It was however only natural that in the great furor of modernizing Mathematics, that enveloped the mathematical community in the first half of the XXth Century would include soon a drive of extending the notion of curvature to (quite) general metric spaces. This tendency came to fruit mainly in the works of Menger [22], Wald [35], [36] and Haantjes [16], [17]. While quickly falling into the desuetude (with very few, but notable exceptions, e.g. [20]), even if efforts [6], [7] were made in correctly re-ascertaining their importance and maintaining their influence, so to say, in other fields, a revival started only towards the end of the previous century and metric curvatures began to reassume their rightful place in Geometry. This is particularly true as far as Menger's curvature is concerned – it has by now become a rather standard and successful tool in Analysis [24], [21], [31] (but not only - see, for instance [12]). In contrast, Haantjes curvature has received in recent literature far lesser attention, with a very few – but notable – exceptions, e.g. [1], where actualized, more elaborate versions of both Menger and Haantjes curvature are discussed in a contemporary framework. On the other hand, Haantjes curvature seems singularly well adapted for a multitude of applicative tasks – see [29], [30], [2], [28] (and perhaps even to applications in such fields as Geometric Group Theory). However, the best candidate for a successful incorporation of the metric metric approach to curvature into the main corpus of contemporary Mathematics, from Differential Geometry to Group Theory, is Wald curvature. Indeed, it turns out that Wald and Alexandrov (comparison) notion of curvature are essentially equivalent (at least for wide range of even mildly-well behaved spaces – see [25] (and also [6])). We tried to emphasize this equivalence and exploit it two our own specific goals in both papers we are summarizing here (and we shall further develop this theme in the the forthcoming lecture notes meant to accompany these proceedings and fully summarize the Colloquium.

## 2. WALD METRIC CURVATURE – A BRIEF OVERVIEW

Wald's approach to the definition of a viable definition of curvature on (quite general) metric spaces was to mimic Gauss' original definition of (total) curvature. However, instead of making appeal to the comparison of infinitesimal areas (which would be unattainable in general metric space not endowed with a measure), he compared quadrangles. Moreover, instead in restricting himself solely to one comparison surface (namely the unit sphere  $S^n$ ), he considered the whole gamut of possible gauge surfaces, namely the surfaces  $S_\kappa$ , where  $S_\kappa$  denotes the complete, simply

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connected surface of constant Gauss curvature  $\kappa$ , i.e.  $S_\kappa \equiv \mathbb{R}^2$ , if  $\kappa = 0$ ;  $S_\kappa \equiv \mathbb{S}_{\sqrt{\kappa}}^2$ , if  $\kappa > 0$ ; and  $S_\kappa \equiv \mathbb{H}_{\sqrt{-\kappa}}^2$ , if  $\kappa < 0$ . Here  $S_\kappa \equiv \mathbb{S}_{\sqrt{\kappa}}^2$  denotes the sphere of radius  $R = 1/\sqrt{\kappa}$ , and  $S_\kappa \equiv \mathbb{H}_{\sqrt{-\kappa}}^2$  stands for the hyperbolic plane of curvature  $\sqrt{-\kappa}$ , as represented by the Poincaré model of the plane disk of radius  $R = 1/\sqrt{-\kappa}$ . We will make this clear in the following sequence of definitions:

**Definition 1.** Let  $(M, d)$  be a metric space, and let  $Q = \{p_1, \dots, p_4\} \subset M$ , together with the mutual distances:  $d_{ij} = d_{ji} = d(p_i, p_j)$ ;  $1 \leq i, j \leq 4$ . The set  $Q$  together with the set of distances  $\{d_{ij}\}_{1 \leq i, j \leq 4}$  is called a *metric quadruple*.

**Definition 2.** The *embedding curvature*  $\kappa(Q)$  of the metric quadruple  $Q$  is defined to be the curvature  $\kappa$  of the gauge surface  $S_\kappa$  into which  $Q$  can be isometrically embedded. (See Figure 1.)

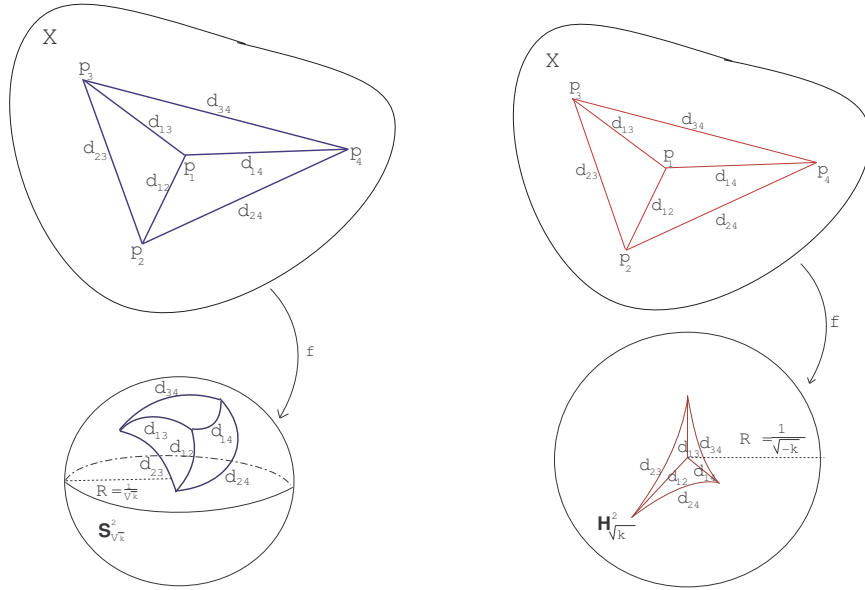


Figure 2.1: Isometric embedding of a metric quadruple in a gauge surface:  $\mathbb{S}_{\sqrt{\kappa}}^2$  (left) and  $\mathbb{H}_{\sqrt{-\kappa}}^2$  (right).

We are now able to bring the definition of *Wald curvature* [35],[36] (or, more precisely, a slight modification of it due to Berestovskii [4]):

**Definition 3.** Let  $(X, d)$  be a metric space. An open set  $U \subset X$  is called a *region of curvature*  $\geq \kappa$  iff any metric quadruple can be isometrically embedded in  $S_m$ , for some  $m \geq \kappa$ .<sup>1</sup>

**Remark 4.** Evidently, in the context of polyhedral surfaces, the natural choice for the set  $U$  required in Definition 3 is the *star* of a given vertex  $v$ , that is, the set  $\{e_{v_j}\}_j$  of edges incident to  $v$ . Therefore, for such surfaces, the set of metric quadruples containing the vertex  $v$  is finite.

Equipped with this quite simple and intuitive choice for  $U$  (and in analogy with Alexandrov spaces – see also [14]) it is quite natural to consider, for *PL* surfaces, the following definition of the Wald curvature  $K(v)$  at the vertex  $v$ :

$$K_W(v) = \min_{v_i, v_j, v_k \in \text{Lk}(v)} K_W^{ijk}(v),$$

where  $K_W^{ijk}(v) = \kappa(v; v_i, v_j, v_k)$ , and where  $\text{Lk}(v)$  denotes the *link* of the vertex  $v$  – see Footnote 2 below.

Here we consider the (intrinsic) *PL* distance between vertices.

<sup>1</sup>While this fact is not needed in the remainder of the paper, we mention for the sake of completeness, that a metric space  $(X, d)$  is said to have *Wald-Berestovskii curvature*  $\geq \kappa$  iff any  $x \in X$  is contained in a region  $U$  of curvature  $\geq \kappa$ .

The definition of the Wald-Berestovskii curvature at an accumulation point of a metric space, hence on a smooth surface, follows naturally by considering limits of the curvatures of (nondegenerate) regions of diameter converging to 0. Moreover, the following important and “reassuring” result holds:

**Theorem 5** (Wald [36]). *Let  $S \subset \mathbb{R}^3$ ,  $S \in \mathcal{C}^m$ ,  $m \geq 2$  be a smooth surface. Then, given  $p \in S$ ,  $\kappa_W(p)$  exists and  $\kappa_W(p) = K(p)$ , where  $K(p)$  denotes the Gaussian curvature at  $p$ .*

A further fact that makes Wald curvature attractive for Discrete Differential Geometry applications is the existence of a concrete (albeit somewhat impractical) computation formula:

Given a metric quadruple  $Q$ ,

$$(2.1) \quad \kappa(Q) = \begin{cases} 0 & \text{if } \Gamma(Q) = 0; \\ \kappa, \kappa < 0 & \text{if } \det(\cosh \sqrt{-\kappa} \cdot d_{ij}) = 0; \\ \kappa, \kappa > 0 & \text{if } \det(\cos \sqrt{\kappa} \cdot d_{ij}) \text{ and } \sqrt{\kappa} \cdot d_{ij} \leq \pi \\ & \text{and all the principal minors of order 3 are } \geq 0; \end{cases}$$

where  $d_{ij} = d(p_i, p_j)$ ,  $1 \leq i, j \leq 4$ , and  $\Gamma(Q) = \Gamma(p_1, \dots, p_4)$  denotes the Cayley-Menger determinant:

$$(2.2) \quad \Gamma(p_0, \dots, p_3) = \begin{vmatrix} 0 & d_{01}^2 & \cdots & d_{13}^2 \\ d_{10}^2 & 0 & \cdots & d_{13}^2 \\ \vdots & \vdots & \ddots & \vdots \\ d_{30}^2 & d_{31}^2 & \cdots & 0 \end{vmatrix}.$$

### 3. METRIC RICCI FLOW FOR PL SURFACES

Our approach to this problem is to pass from the discrete context to the smooth one and explore the already classical results known in this setting, by applying Theorem 5. To this end we have first to make a few observations: One can pass from the *PL* surfaces to smooth ones by employing *smoothings*, defined in the precise sense of *PL* differential Topology (see [23]). Since, by [23], Theorem 4.8, such smoothings are  $\delta$ -*approximations*, and therefore, for  $\delta$  small enough, also  $\alpha$ -approximations of the given piecewise-linear surface  $S_{Pol}^2$ , they approximate arbitrarily well both distances and angles on  $S_{Pol}^2$ . (Due to space restrictions, we do not bring here these technical definitions, but rather refer the reader to [23].) It should be noted that, while Munkres’ results concern *PL* manifolds, they can be applied to polyhedral ones as well, because, by definition, polyhedral manifolds have simplicial subdivisions (and furthermore, such that all vertex links<sup>2</sup> are combinatorial manifolds). Of course, for different subdivisions, one may obtain different polyhedral metrics. However, by the *Hauptvermutung* Theorem in dimension 2 (and, indeed, for smooth triangulations of diffeomorphic manifolds in any dimension, see e.g. [23] and the references therein), any two subdivisions of the same space will be combinatorially equivalent, hence they will give rise to the same polyhedral metric. It follows from the observations above that *metric quadruples* on  $S_{Pol}$  are also arbitrarily well approximated (including their angles) by the corresponding metric quadruples on  $S_m$ . But, by Theorem 5,  $K_{W,m}(p)$  – the *Wald metric curvature* of  $S_m$ , at a point  $p$  – equals the classical (Gauss) curvature  $K(p)$ . Hence the Gauss curvature of the smooth surfaces  $S_m$  approximates arbitrarily well the metric one of  $S_{PL}$  (and, as in [8], the smooth surfaces differ from polyhedral one only on (say) the  $\frac{1}{m}$ -neighbourhood of the 1-skeleton of  $S_{Pol}$ ). Moreover, this statement can be made even more precise, by assuring that the convergence is in the Hausdorff metric. This follows from results of Gromov (see e.g. [30] for details).

We can now introduce the *metric Ricci flow*: By analogy with the classical flow

$$(3.1) \quad \frac{dg_{ij}(t)}{dt} = -2K(t)g_{ij}(t).$$

<sup>2</sup>Recall that the *link*  $\text{lk}(v)$  of a vertex  $v$  is the set of all the faces of  $\overline{\text{St}}(v)$  that are not incident to  $v$ . Here  $\overline{\text{St}}(v)$  denotes the *closed star* of  $v$ , i.e. the smallest subcomplex (of the given simplicial complex  $K$ ) that contains  $\text{St}(v)$ , namely  $\overline{\text{St}}(v) = \{\sigma \in \text{St}(v)\} \cup \{\theta \mid \theta \leq \sigma\}$ , where  $\text{St}(v)$  denotes the *star* of  $v$ , that is the set of all simplices that have  $v$  as a face, i.e.  $\text{St}(v) = \{\sigma \in K \mid v \leq \sigma\}$ .

we define the *metric Ricci flow* by

$$(3.2) \quad \frac{dl_{ij}}{dt} = -2K_i l_{ij},$$

where  $l_{ij} = l_{ij}(t)$  denote the edges (1-simplices) of the triangulation ( $PL$  or piecewise flat surface) incident to the vertex  $v_i = v_i(t)$ , and  $K_i = K_i(t)$  denotes the Wald curvature at the same vertex.

Moreover, we also consider the close relative of (3.1), the normalized flow

$$(3.3) \quad \frac{dg_{ij}(t)}{dt} = (K - K(t))g_{ij}(t),$$

and its metric counterpart

$$(3.4) \quad \frac{dl_{ij}}{dt} = (\bar{K} - K_i)l_{ij},$$

where  $K, \bar{K}$  denote the average classical, respectively Wald, sectional (Gauss) curvature of the initial surface  $S_0$ :  $K = \int_{S_0} K(t)dA / \int_{S_0} dA$ , and  $\bar{K} = \frac{1}{|V|} \sum_{i=1}^{|V|} K_i$ , respectively. (Here  $|V|$  denotes, as usually, the cardinality of the vertex set of  $S_{Pol}$ .)

Before continuing further on, it is important to remark the asymmetry in equation 3.2, that is caused by the fact that the curvature on two different vertices acts, so to say, on the same edge. However, passing to the smooth case, is that the asymmetry in the metric flow that we observed above disappears automatically via the smoothing process. (For further details see [28].)

**3.1. An Approximation Result.** The first result that we can bring is a metric curvature version of classical result of Brehm and Kühnel [8] (where the combinatorial/defect definition of curvature for polyhedral surfaces is used).

**Proposition 6.** *Let  $S_{Pol}^2$  be a compact polyhedral surface without boundary. Then there exists a sequence  $\{S_m^2\}_{m \in \mathbb{N}}$  of smooth surfaces, (homeomorphic to  $S_{Pol}^2$ ), such that*

- (1) (a)  $S_m^2 = S_{Pol}^2$  outside the  $\frac{1}{m}$ -neighbourhood of the 1-skeleton of  $S_{Pol}^2$ ,
- (b) The sequence  $\{S_m^2\}_{m \in \mathbb{N}}$  converges to  $S_{Pol}^2$  in the Hausdorff metric;
- (2)  $K(S_m^2) \rightarrow K_W(S_{Pol}^2)$ , where the convergence is in the weak sense.

**Remark 7.** As we have already noted above, the converse implication – namely that Gaussian curvature  $K(\Sigma)$  of a smooth surface  $\Sigma$  may be approximated arbitrarily well by the Wald curvatures  $K_W(\Sigma_{Pol,m})$  of a sequence of approximating polyhedral surfaces  $\Sigma_{Pol,m}$  – is quite classical.

For a more in-depth discussion and analysis of the convergence rate in the proposition above, see [28].

**3.2. Main Results.** As already stressed, the “good”, i.e. metric and curvature, approximations results quoted, imply that one can study the properties of the metric Ricci flow via those of its classical counterpart, by passing to a smoothing of the polyhedral surface. The use of the machinery of metric curvature considered has the benefit that, by using it, the “duality” between the combinatorics of the packings (and angles) and the metric disappears: The flow becomes purely metric and, moreover, the curvature at each stage (i.e. for every “ $t$ ”) is given – as in the smooth setting – in an intrinsic manner, that is in terms of the metric alone.

We bring here a number of important properties that follow immediately using this approach.

**3.2.1. Existence and Uniqueness.** The main result that we can state here (and, in fact, in this section) is

**Proposition 8.** *Let  $(S_{Pol}^2, g_{Pol})$  be a compact polyhedral 2-manifold without boundary, having bounded metric curvature. Then there exists  $T > 0$  and a smooth family of polyhedral metrics  $g(t), t \in [0, T]$ , such that*

$$(3.5) \quad \begin{cases} \frac{\partial g}{\partial t} = -2K_W(t)g(t) & t \in [0, T]; \\ g(0) = g_{Pol}. \end{cases}$$

(Here  $K_W(t)$  denotes the Wald curvature induced by the metric  $g(t)$ .)

Moreover, both the forwards and the backwards (when existing) Ricci flows have the uniqueness of solutions property, that is, if  $g_1(t), g_2(t)$  are two Ricci flows on  $S_{Pol}^2$ , such that there exists  $t_0 \in [0, T]$  such that  $g_1(t_0) = g_2(t_0)$ , then  $g_1(t) = g_2(t)$ , for all  $t \in [0, T]$ .

Beyond the theoretical importance, the existence and uniqueness of the backward flow would allow us to find surfaces in the conformal class of a given circle packing (Euclidean or Hyperbolic). More importantly, the use of purely metric, Wald curvature based, approach adopted, rather than the combinatorial (and metric) approach of [11], allows us to give a preliminary and purely theoretical answer to Question 2, p. 123, of [11], namely whether there exists a Ricci flow defined on the space of all piecewise constant curvature metrics (obtained via the assignment of lengths to a given triangulation of 2-manifold). Since, by the results of Hamilton [16] and Chow [10], the Ricci flow exists for all compact surfaces, it follows that the fitting metric flow exists for surfaces of piecewise constant curvature. In consequence, given a surface of piecewise constant curvature (e.g. a mesh with edge lengths satisfying the triangle inequality for each triangle), one can evolve it by the Ricci flow, either forward, as in the works discussed above, to obtain, after the suitable area normalization, the polyhedral surface of constant curvature conformally equivalent to it; or backwards (if possible) to find the “primitive” family of surfaces – including the “original” surface obtained via the backwards Ricci flow, at time  $T$  – conformally equivalent to the given one.

**3.2.2. Convergence Rate.** A further type of result, quite important both from the theoretical viewpoint and for computer-driven applications, is that of the *convergence rate* (see [15], [14] for the precise definition).

Since we already know that the solution exists and it is unique (see also the subsection below for the nonformation of singularities), by appealing to the classical results of [16] and [10], we can control the convergence rate of the curvature, as follows:

**Theorem 9.** *Let  $(S_{Pol}^2, g_{Pol})$  be a compact polyhedral 2-manifold without boundary. Then the normalized metric Ricci flow converges to a surface of constant metric curvature. Moreover, the convergence rate is*

- (1) *exponential, if  $\bar{K} = \bar{K}_W < 0$  (i.e.  $\chi(S_{Pol}^2) < 0$ );*
- (2) *uniform; if  $\bar{K} = 0$  (i.e.  $\chi(S_{Pol}^2) = 0$ );*
- (3) *exponential, if  $\bar{K} > 0$  (i.e.  $\chi(S_{Pol}^2) > 0$ ).*

**3.2.3. Singularities Formation.** Another very important aspect of the Ricci flow, both smooth or discrete, is that of singularities formation. Again, a certain (theoretical, at least) advantage of the proposed method presents itself. Indeed, by [11], Theorem 5.1, for compact surfaces of genus  $\geq 2$ , the combinatorial Ricci flow evolves without singularities. However, for surfaces of low genus no such result exists. Indeed, in the case of the Euclidean background metric, that is the one of greatest interest in graphics, singularities do appear. Moreover, such singularities are always combinatorial in nature and amount to the fact that, at some  $t$ , the edges of at least one triangle do not satisfy the triangle inequality. These singularities are removed in heuristic manner. However, by [16], Theorem 1.1, the smooth Ricci flow exists at all times, i.e. no singularities form. From the considerations above, it follows that the metric Ricci flow also exists at all times without the formation of singularities.

**3.2.4. Embeddability in  $\mathbb{R}^3$ .** The importance of the embeddability of the flow is not merely theoretical (e.g. if one considers the problem of the Ricci flow for surfaces of piecewise constant curvature), as it is essential in Imaging (see [3]), and of high importance in Graphics. Indeed, even our very capability of seeing (grayscale) images is nothing but a translation, in the field of vision, of the embeddability of the associated height-surface into  $\mathbb{R}^3$ . (Or, perhaps one should view the mathematical aspect as a formalization of a physical/biological phenomenon...) We should note here that in this respect there exists a certain (mainly theoretical, at this point in time) advantage of our proposed metric flow over the combinatorial Ricci flow [15], [19]. Indeed, in the combinatorial flow, the goal is to produce, via the circle packing metric, a conformal mapping from the given surface to a surface of constant (Gauss) curvature. Since in the relevant cases (see [11]) the surface in question is a planar region (usually a subset of the unit disk), its embeddability (not necessarily isometric) is trivial. Moreover, in the above mentioned works, there is no interest (and indeed, no need) to consider the (isometric) embeddability of the surfaces  $S_t^2$  (see below) for an intermediate time  $t \neq 0, T$ .

The tool that allows us to obtain this type of results is making appeal (again) to  $\delta$ -approximations, in combination with classical results in embedding theory. Indeed, by [23], Theorem 8.8 a  $\delta$ -approximation of an embedding is also an embedding, for small enough  $\delta$ . Since, as we have already mentioned, smoothing represent  $\delta$ -approximations, the possibility of using results regarding smooth surfaces to infer results regarding polyhedral embeddings is proven. (The other direction – namely from smooth to  $PL$  and polyhedral manifolds – follows from the fact that the *secant approximation* is a  $\delta$ -approximation if the simplices of the  $PL$  approximation satisfy a certain nondegeneracy condition – see [23], Lemma 9.3.) We state here the relevant facts:

Let  $S_0^2$  be a smooth surface of positive Gauss curvature, and let  $S_t^2$  denote the surface obtained at time  $t$  from  $S_0^2$  via the Ricci flow. (For all omitted background material – proofs, further results, etc. – we refer to [18].)

**Proposition 10.** *Let  $S_0^2$  be the unit sphere  $\mathbb{S}^2$ , equipped with a smooth metric  $g$ , such that  $\chi(S_0^2) > 0$ . Then the surfaces  $S_t^2$  are (uniquely, up to a congruence) isometrically embeddable in  $\mathbb{R}^3$ , for any  $t \geq 0$ .*

In fact, this results can be slightly strengthened as follows:

**Corollary 11.** *Let  $S_0^2$  be a compact smooth surface. If  $\chi(S_0^2) > 0$ , then there exists some  $t_0 \geq 0$ , such that the surfaces  $S_t^2$  are isometrically embeddable in  $\mathbb{R}^3$ , for any  $t \geq t_0$ .*

In stark contrast with this positive result regarding surfaces uniformized by the sphere, for (complete) surfaces uniformized by the hyperbolic plane we only have the following negative result:

**Proposition 12.** *Let  $(S_0^2, g_0)$  be a complete smooth surface, and consider the normalized Ricci flow on it. If  $\chi(S_0^2) < 0$ , then there exists some  $t_0 \geq 0$ , such that the surfaces  $S_t^2$  are not isometrically embeddable in  $\mathbb{R}^3$ , for any  $t \geq t_0$ .*

#### 4. METRIC RICCI CURVATURE FOR $PL$ MANIFOLDS

We propose a definition of a metric Ricci curvature for  $PL$  manifolds in dimension higher than 2, that does not appeals to smoothings, as in the previous section.

**4.1. The Definition.** While the results in the preceding sections might be encouraging, one would still like to recover in the metric setting a “full” Ricci curvature, namely one that holds for 3 and higher dimensional manifolds, and not just in the degenerate case of surfaces. Our approach (as developed in [14]) is to start from the following classical formula:

$$(4.1) \quad \text{Ric}(e_1) = \text{Ric}(e_1, e_1) = \sum_{i=2}^n K(e_1, e_i).$$

for any orthonormal basis  $\{e_1, \dots, e_n\}$ , and where  $K(e_1, e_j)$  denotes the sectional curvature of the 2-sections containing the directions  $e_1$ .

To adapt this expression for the Ricci curvature to the  $PL$  case, we first have to be able to define (*variational*) *Jacobi fields*. In this we heavily rely upon Stones’s work [32], [33]. Note, however, that we do not need the full strength of Stone’s technical apparatus, only the capability determine the relevant two sections and, of course, to decide what a direction at a vertex of a  $PL$  manifold is.

We start from noting that, in Stone’s work, combinatorial Ricci curvature is defined both for the given simplicial complex  $\mathcal{T}$ , and for its *dual complex*  $\mathcal{T}^*$ . For the dual complex, cells – playing here the role of the planes in the classical setting of which sectional curvatures are to be averaged – are considered. Unfortunately, Stone’s approach for the given complex, where one computes the Ricci curvature  $\text{Ric}(\sigma, \tau_1 - \tau_2)$  of an  $n$ -simplex  $\sigma$  in the direction of two adjacent  $(n - 1)$ -faces,  $\tau_1, \tau_2$ , is not natural in a geometric context (even if useful in his purely combinatorial one), except for the 2-dimensional case, where it coincides with the notion of Ricci curvature in a direction. However, passing to the dual complex will not restrict us, since  $(\mathcal{T}^*)^* = \mathcal{T}$  and, moreover – and more importantly – considering *thick* triangulations enables us to compute the more natural metric curvature for the dual complex and use the fact that the dual of a thick triangulation is thick (for details, see [14]). Recall that thick (also called *fat*) triangulations are defined as follows:

**Definition 13.** Let  $\tau \subset \mathbb{R}^n$ ;  $0 \leq k \leq n$  be a  $k$ -dimensional simplex. The *thickness* (or *fatness*)  $\varphi$  of  $\tau$  is defined as being:

$$(4.2) \quad \varphi(\tau) = \frac{\text{dist}(b, \partial\sigma)}{\text{diam } \sigma},$$

where  $b$  denotes the *barycenter* of  $\sigma$  and  $\partial\sigma$  represents the standard notation for the *boundary* of  $\sigma$  (i.e the union of the  $(n-1)$ -dimensional faces of  $\sigma$ ). A simplex  $\tau$  is  $\varphi_0$ -*thick*, for some  $\varphi_0 > 0$ , if  $\varphi(\tau) \geq \varphi_0$ . A triangulation (of a submanifold of  $\mathbb{R}^n$ )  $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$  is  $\varphi_0$ -*thick* if all its simplices are  $\varphi_0$ -thick. A triangulation  $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$  is *thick* if there exists  $\varphi_0 \geq 0$  such that all its simplices are  $\varphi_0$ -*thick*.

Keeping in mind the notions and facts above, we can now return to the definition of Ricci curvature for simplicial complexes: Given a vertex  $v_0$  in the dual complex, corresponding to a  $n$ -dimensional simplicial complex, a *direction* at  $v_0$  is just an oriented edge  $e_1 = v_0 v_1$ . Since there exist precisely  $n$  2-cells,  $\mathbf{c}_1, \dots, \mathbf{c}_n$ , having  $e_1$  as an edge and, moreover, these cells form part of  $n$  relevant variational (Jacobi) fields (see [32]), the Ricci curvature at the vertex  $v$ , in the direction  $e_1$  is simply

$$(4.3) \quad \text{Ric}(v) = \sum_{i=1}^n K(\mathbf{c}_i),$$

where we define the sectional curvature of a cell  $\mathbf{c}$  in the following manner:

**Definition 14.** Let  $\mathbf{c}$  be a cell with vertex set  $V_{\mathbf{c}} = \{v_1, \dots, v_p\}$ . The *embedding curvature*  $K(\mathbf{c})$  of  $\mathbf{c}$  is defined as:

$$(4.4) \quad K(\mathbf{c}) = \min_{\{i,j,k,l\} \subseteq \{1,\dots,p\}} \kappa(v_i, v_j, v_k, v_l).$$

**Remark 15.** Note that by choosing to work with the dual complex we have restricted ourselves largely to considering solely submanifolds of  $\mathbb{R}^N$ , for some  $N$  sufficiently large. However, in the case of 2-dimensional *PL* manifolds this does not represent restriction, since, by a result of Burago and Zalgaller [9] (see also [27]) such manifolds admit isometric embeddings in  $\mathbb{R}^3$ , embeddings that, furthermore, are unique (up to isometries of the ambient space, of course).

**Remark 16.** Evidently, the definition above presumes that cells in the dual complex have at least 4 vertices. However, except for some utterly degenerate (planar) cases, this condition always holds. Still, even in this case Ricci curvature can be computed using a slightly different approach – see the following remark.

**Remark 17.** It is still possible (by dualization) to compute Ricci curvature according, more or less, to Stone’s ideas, at least for the 2-dimensional case. Indeed, according to [33],

$$(4.5) \quad \text{Ric}(\sigma, \tau_1 - \tau_2) = 8n - \sum_{j=1}^{2n-1} \{|\beta_j| \mid \beta_j < \tau_1 \text{ or } \beta_j < \tau_2; \dim \beta_j = n-2\}.$$

For details and implications of this alternative approach, see [14].

**4.2. Main Results.** The first results one wants to ascertain are those ensuring the convergence of the newly defined Ricci curvature. These are quite straightforward, so here we content ourselves with simply stating them (for further details, see [14]).

**Theorem 18.** Let  $\mathcal{T}$  be a thick simplicial complex, and let  $\mathcal{T}^*$  denote his dual. Then

$$(4.6) \quad \lim_{\text{mesh}(\mathcal{T}) \rightarrow 0} \text{Ric}(\sigma) = \lim_{\text{mesh}(\mathcal{T}^*) \rightarrow 0} C \cdot \text{Ric}^*(\sigma^*),$$

where  $\sigma \in \mathcal{T}$  and where  $\sigma^* \in \mathcal{T}^*$  is (as suggested by the notation) the dual of  $\sigma$ .

**Theorem 19.** Let  $M^n$  be a (smooth) Riemannian manifold and let  $\mathcal{T}$  be a thick triangulation of  $M^n$ . Then

$$(4.7) \quad \text{Ric}_{\mathcal{T}} \rightarrow C_1 \cdot \text{Ric}_{M^n}, \text{ as } \text{mesh}(\mathcal{T}) \rightarrow 0,$$

where the convergence is the weak convergence (of measures).



Beyond these convergence and approximations results, one would like to address deeper issues. Indeed, having introduced a metric Ricci curvature for  $PL$  manifolds, one naturally wishes to verify that this represents a proper notion of Ricci curvature, and not just an approximation of the classical notion. According to the synthetic approach to Differential Geometry, a proper notion of Ricci curvature should satisfy adapted versions of the main, essential theorems that hold for the classical notions. The first and foremost among such theorems is Bonnet-Myers' Theorem and, as expected, fitting versions for combinatorial cell complexes and weighted cell complexes were proven by Stone [32], [33], [34].

In [14] we proved a series of increasingly more general variants of the Bonnet-Myers Theorem, with proofs adapted to the various settings and/or notions of curvature (metric, combinatorial, Alexandrov comparison). Here we bring only two more representative ones.

**Theorem 20** (*PL Bonnet-Myers – metric*). *Let  $M_{PL}^n$  be a complete,  $n$ -dimensional PL, smoothable manifold without boundary, such that*

(i) *There exists  $d_0 > 0$ , such that  $\text{mesh}(M_{PL}^n) \leq d_0$ ;*

(ii)  *$K_W(M_{PL}^n) \geq K_0 > 0$ ,*

*where  $K_W(M_{PL}^n)$  denotes the sectional curvature of the “combinatorial 2-sections”.*

*Then  $M_{PL}^n$  is compact and, moreover*

$$(4.8) \quad \text{diam}(M_{PL}^n) \leq \frac{\pi}{\sqrt{K_0}} .$$

Unfortunately, determining whether a general  $PL$  complex has Wald curvature bounded from below can be, in practice, a daunting task. However, in the special case of thick complexes in  $\mathbb{R}^N$ , for some  $N$  one can determine a simple criterion as follows:

**Theorem 21** (*PL Bonnet-Myers – Thick Complexes*). *Let  $M = M_{PL}^n$  be a complete, connected PL manifold thickly embedded in some  $\mathbb{R}^N$ , such that  $K_W(M^2) \geq K_0 > 0$ , where  $M^2$  denotes the 2-skeleton of  $M$ . Then  $M_{PL}^n$  is compact and, moreover*

$$(4.9) \quad \text{diam}(M_{PL}^2) \leq \frac{\pi}{\sqrt{K_0}} .$$

**4.3. Scalar Curvature and a Comparison Theorem.** Up to this point we were concerned solely with Ricci curvature (as the very title stresses). However, since Ricci curvature is the mean of sectional curvatures we had to consider them too (and, in fact, even more so in view of our definition of Ricci curvature for  $PL$  complexes). We did not discuss, however, scalar curvature. It is only fitting, therefore, for us to add a number of observation regarding this invariant, in particular since a significant result is very easy to formulate and prove.

Of course, we first have to define the scalar curvature  $K_W(M)$  of a  $PL$  manifold  $M$ . In light of our preceding discussion and results, the following definition is quite natural:

**Definition 22.** Let  $M = M_{PL}^n$  be an  $n$ -dimensional  $PL$  manifold (without boundary). The *scalar metric curvature*  $\text{scal}_W$  of  $M$  is defined as

$$(4.10) \quad \text{scal}_W(v) = \sum_{\mathfrak{c}} K_W(\mathfrak{c}),$$

the sum being taken over all the cells of  $M^*$  incident to the vertex  $v$  of  $M^*$ .

**Remark 23.** Observe that the definition of scalar curvature of  $M$  is defined, somewhat counterintuitively, by passing to its dual  $M^*$ . However, this is consistent with our approach to Ricci curvature (and also similar to Stone's original approach – see the discussion in 4.1 above).

From this definition and our previous results (see [14]), we immediately<sup>3</sup> obtain, the following generalization of the classical curvature bounds comparison in Riemannian geometry:

**Theorem 24** (*Comparison theorem*). *Let  $M = M_{PL}^n$  be an  $n$ -dimensional PL manifold (without boundary), such that  $K_W(M) \geq K_0 > 0$ , i.e.  $K(\mathfrak{c}) \geq K_0$ , for any 2-cell of the dual manifold (cell complex)  $M^*$ . Then*

$$(4.11) \quad K_W \leq K_0 \Rightarrow \text{Ric}_W \leq nK_0 .$$

<sup>3</sup>and, in truth rather trivially, since the result holds, regardless of the specific definition for the curvature of a cell.

Moreover

$$(4.12) \quad K_W \underset{\geq}{\cong} K_0 \Rightarrow \text{scal}_W \underset{\geq}{\cong} n(n+1)K_0.$$

**Remark 25.**

(1) Inequality (4.12) can be formulated in the seemingly weaker form:

$$(4.13) \quad \text{Ric}_W \underset{\geq}{\cong} nK_0 \Rightarrow \text{scal}_W \underset{\geq}{\cong} n(n+1)K_0.$$

(2) Note that in all the inequalities above, the dimension  $n$  appears, rather than  $n-1$  as in the smooth, Riemannian case (hence, for instance one has in (4.12),  $n(n+1)K_0$ , instead of  $n(n-1)K_0^4$  as in the classical case). This is due to our definition (4.3) of Ricci (and scalar) curvature, via the dual complex of the given triangulation, hence imposing standard and simple combinatorics, at the price of allowing only for such weaker bounds.<sup>5</sup>

## 5. AN APPLICATION: SMOOTHABLE METRICS ON CUBE COMPLEXES

In this last section we illustrate our belief in the many possible uses of the metric Ricci flow with only (due to space and time restrictions) one example, appertaining to the corpus of “Pure” Mathematics. The following seemingly well known problem in the theory of cube complexes<sup>6</sup> was posed to the author by Joel Haas, together with the basic idea of the first part of the proof, for which the author is deeply grateful.

Let  $\mathcal{C}$  be a cube complex, satisfying the following conditions:

- (1)  $\mathcal{C}$  is negatively curved (i.e. such that  $\#_v Q \geq 4$ , for all vertices  $v$ , where  $\#_v Q$  denotes the number of cubes incident to the vertex  $v$ ;
- (2) The link  $\text{lk}(v)$  of any vertex is a *flag complex*, i.e. a simplicial complex such that any 3-arcs closed curve bounds a triangle (2-simplex), i.e. no such curve separates without being a boundary.<sup>7</sup>

**Question 26.** Does there exist a Riemannian metric  $g$  (on  $\mathcal{C}$ ) such that  $K_g \equiv K$ , where  $K$  denotes the comparison (Alexandrov) curvature of  $\mathcal{C}$ ?

In other words: Does there exist a smoothing of  $(M, g)$  (i.e. Riemannian manifold) of a given cube complex  $\mathcal{C}$  (that has a manifold structure), such that  $K \equiv K_g$ ? Evidently, an important particular case would be that “cubical version of *PL* approximations”), i.e. that of “cubulations” of a (given) Riemannian manifold.

**Remark 27.** The similar problem can be also posed, of course, for positively curved complexes (i.e. such that  $\#_v Q \leq 4$ ). However, we address here only the negatively curved case.

Clearly the answer to Question 26 above is “No”, even if  $\mathcal{C}$  is actually a manifold, since it is not always possible to recover the Riemannian metric from the discrete (“cubical”) one. (Recall that each edge is supposed to be of length 1.) However, in the special case of 3-dimensional cube complexes the question has a positive answer.

We sketch below the proof:

- (1) *Away from the vertices*, i.e. around the edges,<sup>8</sup> one can use a method developed by Gromov and Thurston [13] to produce a generalized type of branched cover (in any dimension). More precisely, (a) construct negatively curved conical surfaces of revolution, with vertex at a vertex  $v$  and with apex angle  $\alpha = 2\pi/n$ , where  $n = \#_v Q$ . Each such cone can be canonically mapped upon a Euclidean cone of apex angles  $\pi/2$ ; then (b) glue the outcome of this process to the result of Step (2) below.

<sup>4</sup>but, on the other hand, this holds even if  $n = 3!$ ...

<sup>5</sup>without affecting the analogue of the Bonnet-Myers Theorem – see Section 2 above.

<sup>6</sup>For a formal definition and more details see, e.g. [26].

<sup>7</sup>Alternatively, this condition may be expressed either as  $\text{lk}(v)$  “has no missing simplices (M. Sageev, [26]), or as “a nonsimplex contains a non edge” (W. Dicks, see [5]).

<sup>8</sup>obviously, in the interiors of the faces the metric is already smooth.

- (2) *Around the vertices* excise an  $\varepsilon$ -ball neighbourhood  $B_\varepsilon$  of  $v$ . On the boundary of  $B_\varepsilon$ , i.e. on the sphere  $S_\varepsilon$  one has the natural triangulation by the intersections of  $S_\varepsilon$  with the cubes of  $\mathcal{C}$  incident with  $v$ . Moreover, the curvature of the vertices of this triangulation will be  $K_\varepsilon \equiv c/\varepsilon^2$ , where  $c$  is some constant. However, while the gluing itself is trivial, one still has to ensure that the result is indeed endowed with a Riemannian metric. For this one has to go through Step 3 of the construction:
- (3) *Smoothen the ball*  $B_\varepsilon$ . In general dimension this represents a daunting problem. Indeed, even in dimension 3, Ricci flow – who represents a natural candidate for smoothing with control of curvature – is yet not attainable, since we can offer, at this point in time, no *PL* (metric) Ricci flow. However, due to Perelman’s resolution of the Poincaré conjecture, in dimension 3 suffices to smoothen the boundary  $S_\varepsilon$ . It is at this point where the method described in Section 3 is applied, producing the required smooth ball  $\hat{S}_\varepsilon$ , that has the same curvature as the *PL*<sup>9</sup> one  $S_\varepsilon$ .

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<sup>9</sup>but not piecewise Euclidean.

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