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Ising fog drip: the shallow puddle, $o(N)$ deep

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Ising fog drip: the shallow puddle, $o(N)$ deep

Dima Ioffe and Senya Shlosman

Abstract

This is an abstract of a work in progress. The first part – *Ising model fog drip: the first two droplets* – is published in "In and Out of Equilibrium 2", Progress in Probability, Birkhauser, 2008.

We study the model of the stalagmite formation. It is a result of the dew-fall effect, when the concentration of the vapor exceeds the saturation point. It turns out that the growth process has discontinuities, when a new atomic monolayer is created spontaneously. An interesting feature of the process is that the size of each newly born monolayer has to exceed some critical size $C_c rN$, where N is the linear size of our 3D system. The study boils down to the investigation of the ensemble of the nested random loops in 2D, which are under the influence of two competing mechanisms: entropic repulsion and weak attraction.

Consider the Ising spins

$$\sigma_t = \pm 1$$

at low temperature β^{-1}, occupying a d-dimensional box V_N^d of the linear size N with $(+)$ boundary conditions. We want to study the sum

$$S_N = \sum_{t \in V_N^d} \sigma_t$$

under the Gibbs distribution μ_N^+.

Let $E_N \approx m^* N^d$ be the expectation of S_N, D_N be its variance, and let us look at the probabilities

$$\Pr\left(S_N = b_N\right)$$

for $b_N < E_N$.

In the Small Deviation Case:

$$\lim_{N\to\infty} \frac{|b_N - E_N|}{\left(N^d\right)^{2/3}} = 0$$

we have

$$P_N\left(b_N\right) = q_N\left(b_N\right)\left(1 + o_N\left(1\right)\right),$$

where

$$q_N\left(b_N\right) = \frac{2}{\sqrt{2\pi D_N}} \exp\left\{-\frac{1}{2}\frac{\left(b_N - E_N\right)^2}{D_N}\right\}.$$

In the Moderate Deviation Case

$$\lim_{N\to\infty} \frac{|b_N - E_N|}{N^d} = 0$$

consider first the case when

$$\lim_{N\to\infty} \frac{|b_N - E_N|}{\left(N^d\right)^{k/k+1}} = 0$$

with $k \leq d$.

Then

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$$P_N(b_N) = q_N(b_N) \times \exp\left\{-\sum_{j=3}^{k} \frac{K_N^j}{j!}\left(\frac{E_N - b_N}{D_N}\right)^j\right\}(1 + o_N(1))$$

$K^3 = -G^3$, $K^4 = -G^4 + 3\frac{(G^3)^2}{D}$, ..., G^j - semi-invariants; $G^3 = \langle\xi^3\rangle - 3\langle\xi^2\rangle\langle\xi\rangle + 2\langle\xi\rangle^3$, ...

In the remaining case

$$\ln P_N(b_N) \sim c(E_N - b_N)^{(d-1)/d}.$$

The reason is that we have reached *the condensation threshold, or dew-point,* which is located at $(N^d)^{d/d+1}$.

It is argued in the paper by Bodineau, Schonmann and Shlosman (3D crystal: how flat its flat facets are? http://fr.arxiv.org/pdf/math-ph/0401010, Communications in Math. Physics. v. 255, n. 3, pp. 747 - 766, 2005.) on heuristic level, that in the low-temperature 3D Ising model in the regime when b_N is already of the volume order, i.e.

$$b_N \sim \nu N^3,$$

the sequence of condensations happens. In such regime one expects to find in the box V_N^3 a droplet Γ of $(-)$-phase, of linear size of the order of N, having the approximate shape of the Wulff crystal, which crystal at low temperatures has 6 flat facets. One expects furthermore that the surface Γ itself has 6 flat facets.

However, when one further increases the "supersaturation parameter" b_N, by an increment of the order of N^2, one expects to observe the condensation of extra $(-)$-particles on one of the flat facets of Γ (randomly chosen), forming a monolayer $\mathfrak{m}$ of thickness of one lattice spacing. So one expects to see here the condensation of the supersaturated gas of $(-)$-particles into a monolayer which is of "visible"size. (Indeed, such monolayers were observed in the experiments of condensation of the Pb.)

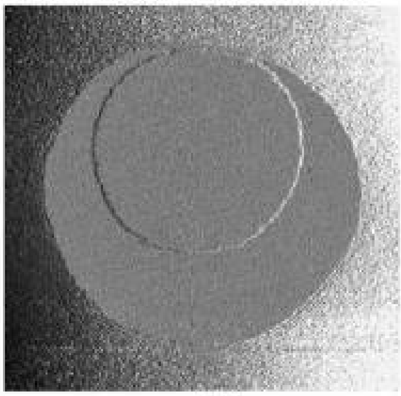

Figure 1: Monolayer of lead.

Suppose we are looking at the Ising spins $\sigma_t = \pm 1$ at low temperature β^{-1} in a 3D box V_N of the linear sizes $RN \times RN \times N$.

We consider the Dobrushin boundary conditions, i.e. we impose $(+)$-boundary conditions in the upper half-space $(z > 0)$, and $(-)$-boundary conditions in the lower half-space $(z < 0)$. These $(\pm)$-boundary conditions force an interface Γ between the $(+)$ and the $(-)$ phases in V_N,

The interface Γ is rigid. The rigidity means that at any location, with probability going to 1 as the temperature $\beta^{-1} \to 0$, the interface Γ coincides with the plane $z = 0$. If we impose the canonical ensemble restriction, fixing the sum S_N to be zero, then the properties of Γ stay the same.

We will now put more -1 particles into V_N; that is, we fix S_N to be

$$S_N = b_N = -\delta N^2,$$

and we will describe the evolution of the surface Γ as the parameter $\delta > 0$ grows.

0.

$$\mathbf{0} \leq \delta < \delta^{\mathbf{1}}$$

Nothing is changed in the above picture – namely, the interface Γ stays rigid. It is essentially flat at $z = 0$; the local fluctuations of Γ are rare and do not exceed $K \ln N$ in linear size.

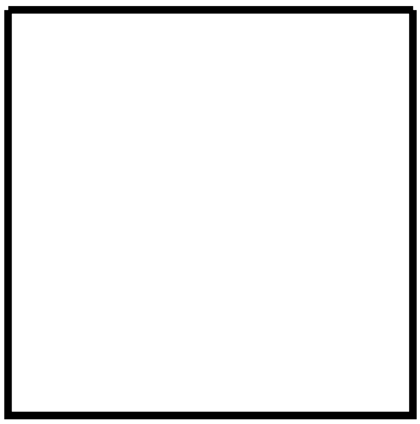

Figure 2: Dry land.

I.

$$\delta^{\mathbf{1}} < \delta < \delta^{\mathbf{2}}$$

The monolayer $\mathfrak{m}_1$ appears on Γ. This is a random outgrowth on Γ, of height one. Inside $\mathfrak{m}_1$ the height of Γ is mostly $z = 1$, while outside it we have mostly $z = 0$. For δ close to δ^1 the shape of $\mathfrak{m}_1$ is *the Wulff shape,* given by the Wulff construction.

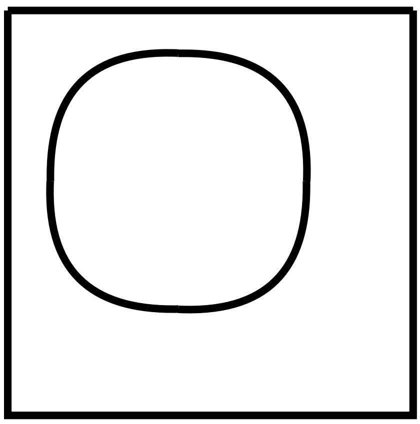

Figure 3: The first droplet is born.

The surface tension function $\tilde{\tau}^{2D}(n)$, $n \in \mathbb{S}^1$, given by

$$\tilde{\tau}(n) = \frac{d}{dn} \tau^{3D}(m) \Big|_{m=(0,0,1)} .$$

Here $\tau^{3D}(m)$, $m \in \mathbb{S}^2$ is the surface tension function of the 3D Ising model, the derivatives are taken at the point $(0, 0, 1) \in \mathbb{S}^2$ along all the tangents $n \in \mathbb{S}^1$ to the sphere $\mathbb{S}^2$. The "radius" of $\mathfrak{m}_1$ is of the order of N, i.e. it equals to $r_1(\delta) N$, and as $\delta \searrow \delta^1$ we have $r_1(\delta) \searrow r_{cr} > 0$. In particular, we never see a monolayer $\mathfrak{m}$ of radius smaller than $r_{cr} N$.

The size r_{cr} scales like $R^{2/3}$. In particular, it is possible to choose R in such a fashion that $R > 2r_{cr}$ or, in other words, for values of R sufficiently large the critical droplet fits into B_N.

As δ increases, the monolayer $\mathfrak{m}_1$ grows in size, and at a certain moment $\delta = \delta^{\mathbf{1.5}}$ it touches the faces of the box B_N. After that moment the shape of $\mathfrak{m}_1$ is different from the Wulff shape. Namely, it is *the Wulff plaquette* , made from four segments on the four sides of the $RN \times RN$ square, connected together by the four quarters of the Wulff shape of radius $\tilde{r}_1(\delta) N$. We have evidently $\tilde{r}_1(\delta^{\mathbf{1.5}}) = R/2$. As $\delta \nearrow \delta^{\mathbf{2}}$, the radius $\tilde{r}_1(\delta)$ decreases to some value $\tilde{r}_1(\delta^{\mathbf{2}}) N$, with $\tilde{r}^1(\delta^{\mathbf{2}}) > 0$.

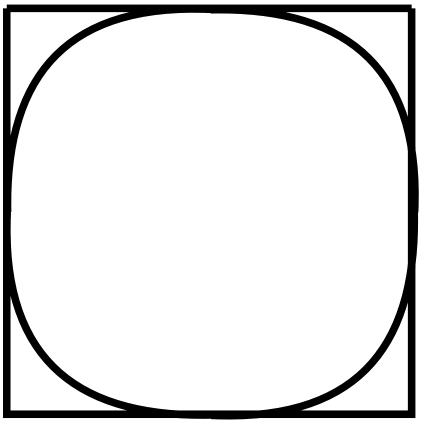

Figure 4: The first droplet meets the wall.

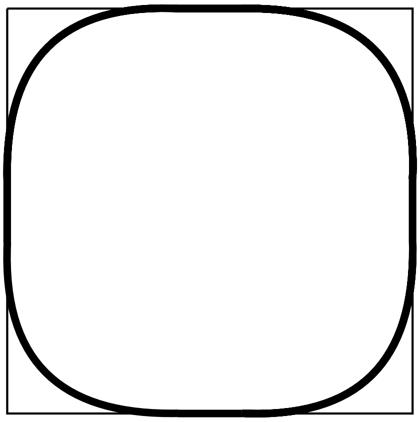

Figure 5: The first droplet is pressed against the wall.

II.

$$\delta^{\mathbf{2}} < \delta < \delta^{\mathbf{2.5}}$$

The second monolayer $\mathfrak{m}_2$ is formed on the top of $\mathfrak{m}_1$. Asymptotically it is of Wulff shape with the radius $r_2(\delta) N$, with $r_2(\delta) \searrow r_2^+(\delta^{\mathbf{2}})$ as $\delta \searrow \delta^2$, with $r_2^+(\delta^2) > 0$. The first monolayer $\mathfrak{m}_1$ has a shape of Wulff plaquette with radius $\tilde{r}_1(\delta)$, which satisfies

$$\tilde{r}_1(\delta) = r_2(\delta).$$

A somewhat curious relation is:

$$r_2^+(\delta^{\mathbf{2}}) \text{ is strictly bigger than } \tilde{r}_1(\delta^{\mathbf{2}}).$$

In other words, the Wulff-plaquette-shaped monolayer $\mathfrak{m}_1$ undergoes a jump in its size and shape as the supersaturation parameter δ crosses the value δ^2. In fact, the monolayer $\mathfrak{m}_1$ shrinks in size: the radius $\tilde{r}_1(\delta)$ increases as δ grows past $\delta^{\mathbf{2}}$.

II.5

$$\delta^{\mathbf{2.5}} < \delta < \delta^{\mathbf{3}}$$

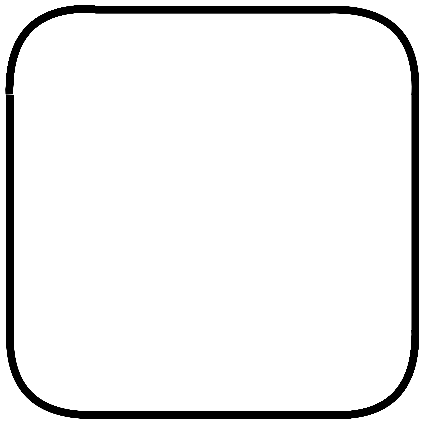

Figure 6: The first droplet is pressed more.

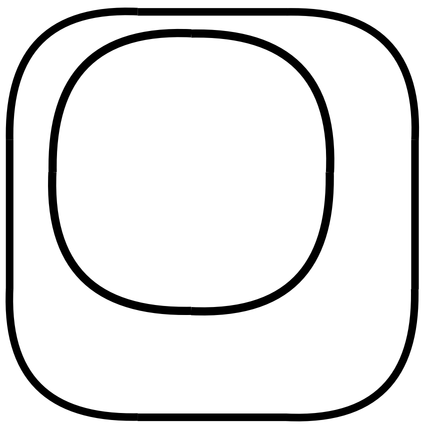

Figure 7: The second droplet is born.

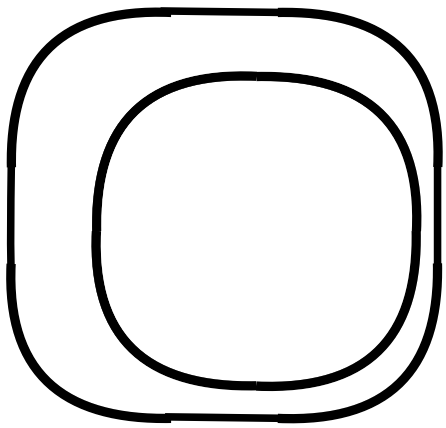

Figure 8: The second droplet is growing, the first is shrinking.

At the value $\delta = \delta^{\mathbf{2.5}}$ the growing monolayer $\mathfrak{m}_2$ meets the shrinking monolayer $\mathfrak{m}_1$, i.e. $r_2\left(\delta^{\mathbf{2.5}}\right) = \tilde{r}_1\left(\delta^{\mathbf{2.5}}\right) = R/2$. Past the value $\delta^{\mathbf{2.5}}$ the two monolayers $\mathfrak{m}_2 \subset \mathfrak{m}_1$ are in fact asymptotically equal, both having the shape of the Wulff plaquette with the same radius $\tilde{r}_1(\delta) = \tilde{r}_2(\delta)$, decreasing to the value $\tilde{r}_1\left(\delta^{\mathbf{3}}\right) = \tilde{r}_2\left(\delta^{\mathbf{3}}\right)$ as δ increases up to $\delta^{\mathbf{3}}$.

III.

$\delta^{\mathbf{3}} < \delta < \delta^{\mathbf{4}}$

The third monolayer $\mathfrak{m}_3$ is formed, of the asymptotic radius $r_3(\delta)N$, with $r_3(\delta) \searrow r_3^+(\delta^{\mathbf{3}})$ as $\delta \searrow \delta^{\mathbf{3}}$, with $r_3^+(\delta^{\mathbf{3}}) > 0$.

The radii of two bottom Wulff plaquettes $\tilde{r}_1(\delta) = \tilde{r}_2(\delta) = r_3(\delta)$ decrease to the value $r_3^+(\delta^{\mathbf{3}})$ as δ decreases down to $\delta^{\mathbf{3}}$, with $r_3^+(\delta^{\mathbf{3}}) > \tilde{r}_i(\delta^{\mathbf{3}})$, so the two Wulff plaquettes $\mathfrak{m}_1, \mathfrak{m}_2$ shrink, jumping to a smaller area, as δ passes the threshold value $\delta^{\mathbf{3}}$.

...

X.

At a certain step k the structure of the jump changes. The newly born monolayer is a Wulff plaquette, having the same size as all the lower lying Wulff plaquettes. They grow together until the moment when the $k+1$-th layer is born, at which moment all k Wulff plaquettes shrink a bit.

In the language of the interacting random walks our problem looks as follows. We have a collection of random loops γ_i, $i = 1, 2, ..., k$, with the total area $A(\gamma_1) + ... + A(\gamma_k) = S$, $k = k(S)$. The loops pay heavily if they intersect, or if they have excess length. Therefore they do not want to come too close to each other because of the entropic repulsion. But on the other hand there is an attractive interaction between the paths.

The weight w of the family $G_k = \{\gamma_i\}$ is given by

$$w(G_k) = \sum_{\Gamma^n} \exp\left\{-\beta \sum_{i=1}^{k} |\gamma_i| + \sum_{\Lambda \subset \mathbb{Z}^2} \Phi(\Lambda) N(\Lambda, G_k)\right\}.$$

The interaction part satisfies

$$|\Phi(\Lambda)| \leq \exp\{-\beta diam(\Lambda)\}.$$

Here $N(\Lambda, G_k)$ is the number of the paths γ_i in G_k which intersect Λ, lessen by 1.

It turns out that if the number k of loops in G_k growth slower than N, i.e. is $k = o(N)$, then the entropic repulsion wins over. The loops do not interact too much, except that they have to remain nested. The area inside the loops is the same on the scale N^2. So macroscopically the loops are indistinguishable, i.e. from far away one sees just a single contour.

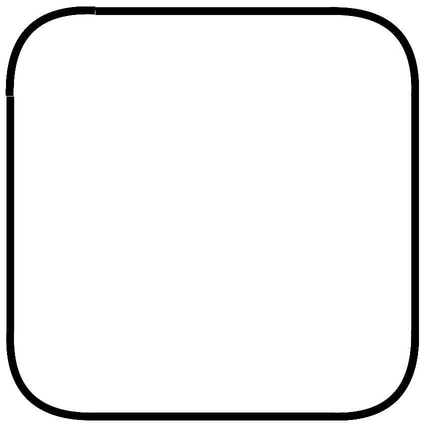

Figure 9: $o(N)$ nested loops.

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