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Irreducibility of ideals in a one-dimensional analytically irreducible ring

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Abstract

Let \( R \) be a one-dimensional analytically irreducible ring and let \( I \) be an integral ideal of \( R \). We study the relation between the irreducibility of the ideal \( I \) in \( R \) and the irreducibility of the corresponding semigroup ideal \( v(I) \). It turns out that if \( v(I) \) is irreducible, then \( I \) is irreducible, but the converse does not hold in general. We collect some known results taken from [5], [4], [3] to obtain this result, which is new. We finally give an algorithm to compute the components of an irredundant decomposition of a nonzero ideal.

A numerical semigroup is a subsemigroup of \( \mathbb{N} \), with zero and with finite complement in \( \mathbb{N} \). The numerical semigroup generated by \( d_1, \ldots, d_v \in \mathbb{N} \) is \( S = \langle d_1, \ldots, d_v \rangle = \{ \sum_{i=1}^{v} n_i d_i, n_i \in \mathbb{N} \} \). \( M = S \setminus \{0\} \) will denote the maximal integer of \( S \), \( e \) the multiplicity of \( S \), that is the smallest positive integer of \( S \), \( g \) the Frobenius number of \( S \), that is the greatest integer which does not belong to \( S \). A relative ideal of \( S \) is a nonempty subset \( I \) of \( \mathbb{Z} \) such that \( I + S \subseteq I \) and \( I + s \subseteq S \), for some \( s \in S \). A relative ideal which is contained in \( S \) is an integral ideal of \( S \). If \( I, J \) are relative ideals of \( S \), then the following are relative ideals too: \( I \cap J, I \cup J, I + J = \{ i + j, i \in I, j \in J \}, I - J = \{ z \in \mathbb{Z} | z + J \subseteq I \}, I - J = (I - J) \cap S \). An integral ideal \( I \) of a numerical semigroup \( S \) is called irreducible if it is not the intersection of two integral ideals which properly contain \( I \). Consider the partial order on \( S \) given by \( s_1 \leq s_2 \iff s_1 + s_3 = s_2 \), for some \( s_3 \in S \), and for \( s \in S \), set \( B(s) = \{ x \in S | x \leq s \} \).

**Proposition 1.** Let \( I \) be a proper integral ideal of \( S \). Then \( I \) is irreducible if and only if \( I = S \setminus B(s) \), for some \( s \in S \).

**Theorem 1.** a) If \( I \) is a proper integral ideal of \( S \) and if \( (I - M) \setminus I = \{ s_1, \ldots, s_n \} \), then \( I = (S \setminus B(s_1)) \cap \ldots \cap (S \setminus B(s_n)) \) is the unique irredundant decomposition of \( I \) in integral irreducible ideals.

b) \( I \) is irreducible if and only if \( |(I - M) \setminus I| = 1 \).

A relative ideal \( I \) of a numerical semigroup \( S \) is called \( \mathbb{Z} \)-irreducible if it is not the intersection of two relative ideals which properly contain \( I \). A particular relative ideal of \( S \) plays a special role, it is the canonical ideal \( \Omega \) which is maximal with respect to the property of non containing \( g \), the Frobenius number of \( S \). Thus \( \Omega = \{ g - x, x \in \mathbb{Z} \setminus S \} \).

**Proposition 2.** Let \( J \) be a relative ideal of \( S \). Then \( J \) is \( \mathbb{Z} \)-irreducible if and only if \( J = \Omega + z \) for some \( z \in \mathbb{Z} \), if and only if \( |(J - \Omega \setminus M) \setminus J| = 1 \).

**Theorem 2.** \( I \) is a relative ideal of \( S \) minimally generated by \( i_1, \ldots, i_h \) if and only if \( \Omega - I = (\Omega - i_1) \cap \ldots \cap (\Omega - i_h) \) is the unique irredundant decomposition of \( \Omega - I \) in \( \mathbb{Z} \)-irreducibles ideals.

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Corollary 1. Each relative ideal $J$ of $S$ has a unique irredundant decomposition as intersection of $\mathbb{Z}\text{-irreducible}$ ideals. The number of components is the cardinality of a minimal set of generators of $\Omega = \frac{J}{Z}$, which is also equal to $|\{J + M\}/J|$.

Applications to one-dimensional Rings: As usual, an integral ideal $I$ of a ring $R$ is called irreducible if it is not the intersection of two proper overideals. A fractional ideal $F$ of a ring $R$ with total ring of fractions $K$ is called $K\text{-irreducible}$ if it is not the intersection of two strictly larger fractional ideals.

Using the following lemma, we recover a known result with Proposition 3 below [4, Proposition 3.1.6, p.67].

Lemma 1. Let $I$ be an ideal of a local ring $(R,m)$ and $J$ an irreducible ideal such that $I \subseteq J$. Then $l_R((I : \overline{m})/J \cap (I : \overline{m})) \leq 1$.

Proposition 3. Let $(R,m)$ be a Noetherian local ring and $I$ be an $m$-primary ideal. Then the number $n(I)$ of components of an irredundant decomposition of $I$ is

$$n(I) = l_R((I : \overline{m})/I) = \dim_{R/I} \text{Socle}(R/I)$$

Corollary 2. Let $(R,m)$ be a Noetherian local ring and $I$ be an $m$-primary ideal. Then $I$ is irreducible if and only if $l_R((I : \overline{m})/I) = 1$.

Let $R$ be an integral domain with field of fractions $K$. A fractional ideal $\omega$ of $R$ is called an $m$-canonical ideal if for any nonzero fractional ideal $I$ of $R$, we have $I = \omega : \overline{R} (\omega : I)$. We fix from here on the following notation: $(R,m)$ is a one-dimensional analytically irreducible Noetherian domain. This is a domain for which the integral closure $V = \overline{R}$ in the field of fractions $K$ of $R$ is a rank-one discrete valuation domain and is a finitely generated $R$-module.

Let $v : K \setminus \{0\} \rightarrow \mathbb{Z}$ be the normalized valuation associated to $V$. Thus, if $t \in V$ generates the maximal ideal of $V$, then $v(t) = 1$. Moreover, we assume that $R/m \simeq V/M$, where $M = tV$ is the maximal ideal of $V$, i.e. $R$ is residually rational. A one-dimensional analytically irreducible Noetherian domain has an $m$-canonical ideal, cf. e.g. [2]. Observe that: $S = v(R) = \{v(r) : r \in R \setminus \{0\}\}$ is a numerical semigroup. We denote by $\Omega$ the canonical ideal of $v(R)$.

Proposition 4. Let $F$ be a fractional ideal of $R$. Then $F$ is $K\text{-irreducible}$ if and only if $l_R(F : \overline{m/F}) = 1$ if and only if $v(F) = \Omega + z$, for some $z \in \mathbb{Z}$.

Corollary 3. Let $F$ be a fractional ideal of $R$. Then $F$ is $K\text{-irreducible}$ if and only if $v(F)$ is $\mathbb{Z}\text{-irreducible}$.

It is a natural question to ask whether a result similar to Corollary 3 holds for integral ideals.

Theorem 3. Let $I$ be a non-zero integral ideal of $R$ such that $v(I)$ is irreducible, then $I$ is irreducible.

Proof: Now $I$ is $m$-primary, so $I \subset (I : \overline{m})$. Since $v(I : \overline{m}) \leq (v(I) - v(m)) \leq (v(I) - v(m)) \leq 1$, where the last equality follows from Theorem 1 b). So by Corollary 2, $I$ is irreducible.

The converse of Theorem 3 does not hold, as the following example shows.

Example: $S = \{2,5\}$, $R = k[[t^2,t^5]]$, $I = (t^4+t^5+t^7)$, we have: $v(I) = v(m) = 0.5$, $v(S) = 5$, then by Theorem 1, $v(I) = (S \setminus B(2)) \cap (S \setminus B(5))$. So $v(I)$ is not irreducible.

But, $I = (I : \overline{m}/I) = (v(I) : \overline{m}/I) = 1$. In fact, consider $f = a_2t^2 + a_4t^4 + a_5t^5 + \ldots \in R$, with $a_2 \neq 0$. If $f \notin I$, then $f \notin I$, such that $h = b_0 + b_2t^2 + b_4t^4 + \ldots$, then $0 = b_0 = a_2$, so that $f \notin I$. Thus $f \notin (I : \overline{m})$. Hence $I$ is irreducible.

Corollary 4. Let $I$ be a monomial ideal of $k[[t^1,\ldots,t^m]]$. Then, $I$ is irreducible if and only if $v(I)$ is irreducible.
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In other terms, the non trivial deduction of Corollary 4 says that, if \( I \) is a monomial ideal which is not the intersection of two strictly larger monomial ideals, then \( I \) is not the intersection of two strictly larger ideals, even if non monomial ideals are allowed. This is indeed known in a more general context [6, Proposition 11, p.41].

**Algorithm:** The following algorithm is a method for computing the components of an irredundant decomposition of a non zero ideal \( I \) of \( R \).

1. Compute the length of \( (I :_R m/I) \) as \( R \)-module, \( l_R(I :_R m/I) = n \)
2. Look at a set of generators of \( (I :_R m/I) \) as \( R/m \) vector space.
\[
(I :_R m/I) = \langle f_1 + I, \ldots, f_n + I \rangle.
\]
3. Let for \( i = 1, \ldots, n \),
\[
J_i = (I, f_1, f_{i-1}, f_{i+1}, \ldots, f_n)
\]
4. For each \( J_i \), we will construct another ideal \( J'_i \) such that \( J_i \subseteq J'_i \), \( J'_i \) is irreducible and \( \bigcap J'_i = I \).
5. Compute the length of \( (J_i :_R m/J_i) \).
   If \( l_R(J_i :_R m/J_i) = 1 \), then we take \( J'_i = J_i \).
   If not, look at a set of generators \( (g_1 + J_i, \ldots, g_j + J_i) \) of \( (J_i :_R m/J_i) \) as \( R/m \) vector space.
   Since \( I :_R m \subseteq J_i :_R m \) and \( f_i \notin J_i \), we can take \( g_1 = f_i \).
6. Iterating the construction above we will obtain:
\[
J_{i_1} = (J_i, g_2, \ldots, g_s)
\]
\[
J_{i_2} = (J_i, f_1, g_3, \ldots, g_s)
\]
\[
\vdots
\]
\[
J_{i_k} = (J_i, f_1, \ldots, g_{s-1})
\]
Yet we are interested only in the ideal \( J_{i_1} \) which does not contain \( f_i = g_1 \).
7. Compute the length of \( (J_{i_1} :_R m/J_{i_1}) \).
   If \( l_R(J_{i_1} :_R m/J_{i_1}) = 1 \), then we take \( J'_i = J_{i_1} \). If not we proceed in the same way. After at most \( k - 2 \) steps, where \( k = l_R(R/I) \), we find an irreducible ideal \( J'_i \).
8. It is easy to see that \( \bigcap_{i \neq j} J'_i \not\subseteq J'_i \), because \( f_i \in \bigcap_{i \neq j} J'_i \) but \( f_i \notin J'_i \). We claim that \( I = \bigcap_{i=1}^n J'_i \) is an irreducible intersection of \( I \) into irreducible ideals. In fact suppose that we have \( I \subseteq \bigcap_{i=1}^n J'_i \). Then
\[
I \subseteq \bigcap_{i=1}^n J'_i \subseteq J'_1 \cap \ldots \cap J'_n \subseteq \ldots \subseteq J'_n \cap J'_{n-1} \cap \ldots \cap J'_2 \cap J'_1 \cap \left( (I :_R m) \cap (I :_R m) \right),
\]
a contradiction since \( l_R = (I :_R m/I) = n \), so \( I = \bigcap_{i=1}^n J'_i \).

**References**


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