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The category of cofinite modules for ideals of dimension one and codimension one

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We assume that all rings are commutative and noetherian with identity throughout this paper. In this paper, we shall introduce several results on the category \( \mathcal{M}(A, I)_{\text{cof}} \) (See Definition 1 below) for ideals \( I \) of dimension one and codimension one (cf. [11] and [9]).

1. Introduction

In this section, we introduce former results on our research and several definitions. The following theorem is fundamental, due to Matlis and Grothendieck (cf. [13] and [3]).

**Theorem A.** Let \( A \) be a complete local ring, with maximal ideal \( m \), and residue field \( k = A/m \). Let \( E = E_A(k) \) be an injective hull of \( k \) over \( A \). For an \( A \)-module \( N \), the following conditions are equivalent.

1. \( N \) satisfies the descending chain conditions (dcc);
2. \( N \) is a submodule of \( E^n \), the direct sum of \( n \) copies of \( E \), for some \( n \);
3. There is an \( A \)-module \( M \) of finite type such that \( N \) is isomorphic to \( \text{Hom}_A(M, E) \);
4. \( \text{Supp}_A N \subseteq V(m) \) and \( \text{Hom}_A(k, N) \) is of finite type;
5. \( \text{Supp}_A N \subseteq V(m) \) and \( \text{Ext}^i_A(k, N) \) is of finite type for all \( i \);
6. \( \text{Supp}_A N \subseteq V(m) \) and \( \text{Hom}_A(N, E) \) is of finite type.

**Proof.** See [5] for the proof (See [8] also).

Next recall several definitions. Let \( \mathcal{M}(A) \) be the category of all modules over a ring \( A \).

**Definition 1 (I-cofiniteness on modules).** Let \( \mathcal{M}(A, I)_{\text{cof}} \) be the class of modules \( N \) over a ring \( A \) satisfying the condition

\( \text{Supp}_A(N) \subseteq V(I) \) and \( \text{Ext}^j_A(A/I, N) \) is of finite type, for all \( j \),

where \( I \) is an ideal of \( A \). The objects of \( \mathcal{M}(A, I)_{\text{cof}} \) are called I-cofinite.

**Definition 2 (Abelian category).** Let \( A, I \) and \( \mathcal{M} = \mathcal{M}(A, I)_{\text{cof}} \) be as above. The full subcategory \( \mathcal{M} \) is called Abelian, if it is closed under the kernel and cokernel of a morphism (See [6, p. 202] for the definition of Abelian category).

**Definition 3 (Derived categories and Thick subcategories (cf. [7] and [12])).** Let \( D^+(A) \) be the derived category, whose objects are complexes consisting of \( A \)-modules, where we write \( ^+ \) in place of \( +, -, b \) or \( b \). Further let \( A' \) be a thick Abelian subcategory of \( \mathcal{M}(A) \), that is any extension in \( \mathcal{M}(A) \) of two objects of \( A' \) is in \( A' \). We define \( D^+_{A'}(A) \) to be the full subcategory of \( D^+(A) \)
consisting of those complexes $X^\bullet$ whose cohomology objects $H^i(X^\bullet)$ are all in $A'$. In this paper, we denote $\mathcal{D}_{ft}^r(A)$ for $\mathcal{D}_n^r(A)$ in the case that $A'$ is the category consisting of all $A$-modules of finite type, following the notations of [5].

**Definition 4** (I-dualizing functor). Let $A$ be a ring equipped with a dualizing complex $\mathbf{D}$, $I$ an ideal of $A$. Let $\Gamma_I(-)$ be the $I$-power torsion subfunctor of the identity functor on $\mathcal{M}(A)$ (cf. [12, §1]). Set $D_I(-)$ to be the functor $\mathbb{R} \text{Hom}^*\left(-, \mathbb{R}\Gamma_I(\mathbf{D})\right)$ on the derived category $\mathcal{D}(A)$. In this paper, we call this functor $D_I(-)$ the I-**dualizing functor** (See [12, § 4.3]).

**Definition 5** (I-cofiniteness on complexes). Let $A$ and $I$ be as above. Let $N^\bullet$ be an object of the derived category $\mathcal{D}(A)$. We say that $N^\bullet$ is I-**cofinite**, if there exists $M^\bullet \in \mathcal{D}_{ft}(A)$, such that $N^\bullet \cong D_I(M^\bullet)$ in $\mathcal{D}(A)$. Here $D_I(-)$ is the I-dualizing functor.

Here we recall the affine duality theorem and a characterization of cofinite complexes (See [5] for the proofs):

**Theorem B** (Affine duality theorem). Let $R$ be a regular ring of finite Krull dimension $d$ and $J$ an ideal of $R$. Suppose that $R$ is complete with respect to $J$-adic topology. Then the natural morphism of functors $id \to D_J \circ D_J$ is an isomorphism, for complexes in either of the categories $\mathcal{D}_{ft}(R)$ or $\mathcal{D}(R, J)_{cof}$, where we denote by $\mathcal{D}(R, J)_{cof}$ the essential image of $\mathcal{D}_{ft}(R)$ by $D_J(-)$.

**Theorem C** (Characterization of cofinite complexes). Let $R$ and $J$ be as above, $N^\bullet \in \mathcal{D}(R)$. Suppose that $R$ is complete with respect to the $J$-adic topology. Then $N^\bullet$ is $J$-cofinite if and only if

\begin{itemize}
  \item[(a)] $\text{Supp} H^i(N^\bullet) \subseteq V(J)$ for each $i$, and
  \item[(b)] $\text{Ext}^i(R/J, N^\bullet)$ is of finite type over $R$, for each $j$.
\end{itemize}

It is natural to ask whether Theorem A holds for non-maximal ideals of $A$. Four questions were proposed in the paper [5, §2]. In particular the following are given:

**Question 1** (Second Question). Let $J$ be an ideal of a regular ring $R$ of finite Krull dimension. Does the class $\mathcal{M}(R, J)_{cof}$ form an Abelian full subcategory of $\mathcal{M}(R)$?

**Question 2** (Fourth Question). Does there exist an Abelian category $\mathcal{M}_{cof}$ consisting of $R$-modules, such that objects $N^\bullet \in \mathcal{D}(R, J)_{cof}$ are characterized by the property “$H^i(N^\bullet) \in \mathcal{M}_{cof}$” for all $i$?

In [5, §3 An Example], Question 1 and Question 2 are answered negatively for an ideal of dimension two. The example is as follows: Let $R$ be the formal power series ring $k[[u, v]]$ over a polynomial ring $k[x, y]$, where $k$ is a field and $J$ the ideal $(u, v)$ of $R$. Let $M$ be the $R$-module $R/(xv + yu)$. Then it is proved that the local cohomology module $H^2_\mathfrak{m}(M)$ is not $J$-cofinite in [5, §3 An Example]. Even the socle $\text{Hom}_R(k, H^2_\mathfrak{m}(M))$ is not finitely generated. The ideal $J$ is of dimension two and not principal, and there is an exact sequence:

$$0 \rightarrow H^1_\mathfrak{m}(M) \rightarrow H^2_\mathfrak{m}(R) \rightarrow H^2_\mathfrak{m}(R) \rightarrow H^2_\mathfrak{m}(M) \rightarrow 0.$$ 

Since $J$ is generated by a regular sequence $u, v$, the local cohomology module $H^2_\mathfrak{m}(R)$ is $J$-cofinite. If Question 1 is affirmatively answered for the ideal $J$, then the local cohomology module $H^2_\mathfrak{m}(M)$ must be $J$-cofinite, which is false for this example. Further, if Question 2 is affirmatively answered for the ideal $J$, then $\text{Hom}_R(R/J, H^2_\mathfrak{m}(M))$ must be of finite type by the local duality theorem (cf. [5, Theorem 2.1]) and the characterization of cofinite complexes, which gives a contradiction.

2. THE CASES FOR IDEALS OF DIMENSION ONE OVER LOCAL RINGS

Now we shall introduce the following theorems:

**Theorem 1** (cf. [11, Theorem 1]). Let $(A, \mathfrak{m})$ be a local ring, and $I$ an ideal of $A$. If $I$ is an ideal of $A$ of dimension one, then $\mathcal{M}(A, I)_{cof}$ is an Abelian full subcategory of $\mathcal{M}(A)$.

**Theorem 2** (cf. [11, Theorem 2]). Let $(R, \mathfrak{n})$ be a regular local ring, and $J$ an ideal of $R$ of dimension one. Let $N^\bullet$ be in the derived category $\mathcal{D}(R)$ and suppose that $R$ is complete with respect to the $J$-adic topology. Then $N^\bullet$ is $J$-cofinite if and only if $H^i(N^\bullet)$ is in $\mathcal{M}(R, J)_{cof}$ for all $i$.
Remark 1. Recently Theorem 2 is extended to complete Gorenstein domains, using the refined Lemmas from those of Huneke-Koh [8] (cf. [1, Theorem 1]).

Delfino and Marley proved that $\mathcal{M}(A, P)_{\text{cof}}$ is an Abelian full subcategory of $\mathcal{M}(A)$ for a prime ideal $P$ of dimension one over a complete local ring $A$ (cf. [2, Theorem 2]). Melkersson proved some related results (cf. [14, Theorem 7.4, Theorem 7.6, Theorem 7.7]).

3. THE CASES FOR IDEALS OF CODIMENSION ONE OVER RINGS

The following result from [9] may have been known before, though the author has been unable to find it in the literature.

Theorem 3 (cf. [9]). Let $A$ be a noetherian ring, and $I$ an ideal of $A$. If $I$ is an ideal generated by one element $x$ of $A$ up to radical, then $\mathcal{M}(A, I)_{\text{cof}}$ is an Abelian full subcategory of $\mathcal{M}(A)$.

Remark 2. Let $M$ be a non zero module in $\mathcal{M}(A, I)_{\text{cof}}$. If $\sqrt{x} = \sqrt{(r)}$ and $x$ is not a unit, then $x^n$ is a zero divisor on $M$ for some $n$, since $\text{Supp} M$ is contained in $V(x)$. Further it holds that $\Gamma_I(M) = M$.

The following also holds from Theorem 3, since the height one prime ideal is principal in a unique factorization domain.

Corollary 1. Let $R$ be a unique factorization domain, and $J$ an ideal of pure height one. Then $\mathcal{M}(R, J)_{\text{cof}}$ is an Abelian full subcategory of $\mathcal{M}(R)$.

Finally, the author conjectures that Theorem 1 may be true without the hypothesis that the ring be local, though this has not yet been proved:

Conjecture. Let $A$ be a noetherian ring, which is not local, and $I$ an ideal of $A$. If $I$ is an ideal of dimension one, then the category $\mathcal{M}(A, I)_{\text{cof}}$ is Abelian.

On the other hand, the author suspects that $\mathcal{M}(A, I)_{\text{cof}}$ is a Serre subcategory of $\mathcal{M}(A)$, for an ideal $I$ of dimension one. But he has no counterexample.

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References


